A Complex Analysis Approach to the Motion of Uniform Vortices

Giorgio Riccardi

Dept. of Mathematics and Physics, University of Campania "*Luigi Vanvitelli*"
viale Lincoln, 5 - 81100 Caserta, Italy *and*

CNR-INSEAN, National Research Council of Italy - Maritime Research Center
via di Vallerano 139, 00128 Rome, Italy

email: giorgio.riccardi@unicampania.it

1 Introduction

The investigation of the inviscid, planar dynamics of uniform vortices by means of the Complex Analysis dates back to the last two decades of the twentieth-century [1], when the velocity induced by a uniform (bounded) vortex has been related to the Schwarz function of its (bounded, smooth) boundary. The vortex shape was assumed nearly circular and the analysis took advantage of the Laurent series representation of the boundary. Furthermore, the induced velocity was obtained by writing the Schwarz function as a sum of two functions, one analytical inside and the other outside the vortex [2]. At later times, similar approaches have been used for investigating vortex equilibria and their stabilities [3, 4, 5], by means of an *ad hoc* relation between Schwarz function and streamfunction in a quiescent fluid.

The analytical approach to the vortex dynamics does not received the same attention, basically for the lack of a general relation between Schwarz function and induced velocity. The present paper summarizes a research line undertaken by the author in the last ten years [6, 7, 8, 9, 10] and based on a novel, general relation between Schwarz function and velocity.

2 New approaches to vortex flows by means of Complex Analysis

An integral relation between the Schwarz function $\boldsymbol{\Phi}$ of the boundary of a uniform vortex P and the (conjugate) induced velocity has beeen deduced in [7]:

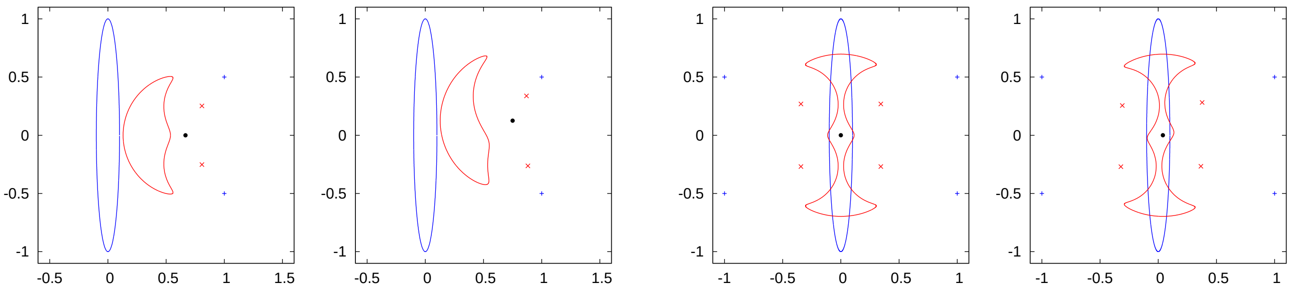
$$\overline{\boldsymbol{u}}(\boldsymbol{x};t) = \frac{\omega}{2\boldsymbol{i}} \left[\chi_{P(t)}(\boldsymbol{x})\ \boldsymbol{\Phi}(\boldsymbol{x};t) - \frac{1}{2\pi\boldsymbol{i}} \int_{\partial P(t)} d\boldsymbol{y}\ \frac{\boldsymbol{\Phi}(\boldsymbol{y};t)}{\boldsymbol{y}-\boldsymbol{x}} \right] , \tag{1}$$

where ω is the vorticity level (uniform inside P and constant in time) and χ_P holds 0 outside P, 1 inside and just 1/2 on the boundary. Some applications of this relation are now briefly described.

A vortex is in equilibrium if a rotating (at angular speed Ω) frame of reference exists, in which the shape of the vortex is steady, *i.e.* the relative velocity is tangent to the vortex boundary. The equilibrium can be achieved also in presence of an external velocity field $\omega\boldsymbol{F}(\boldsymbol{x};t)/(2\boldsymbol{i})$ due, for example, to stationary (in the rotating frame) point vortices. Once the relation (1) is used (with $\boldsymbol{C}(\boldsymbol{\Phi})$ in place of the Cauchy integral, for short) the condition on the velocity leads to the eigenvalue problem:

$$\mathrm{Re}\left\{\left[(\lambda - 1/2)\boldsymbol{\Phi} + \boldsymbol{C}(\boldsymbol{\Phi}) - \boldsymbol{F}\right] \boldsymbol{\tau}\right\} = 0 , \tag{2}$$

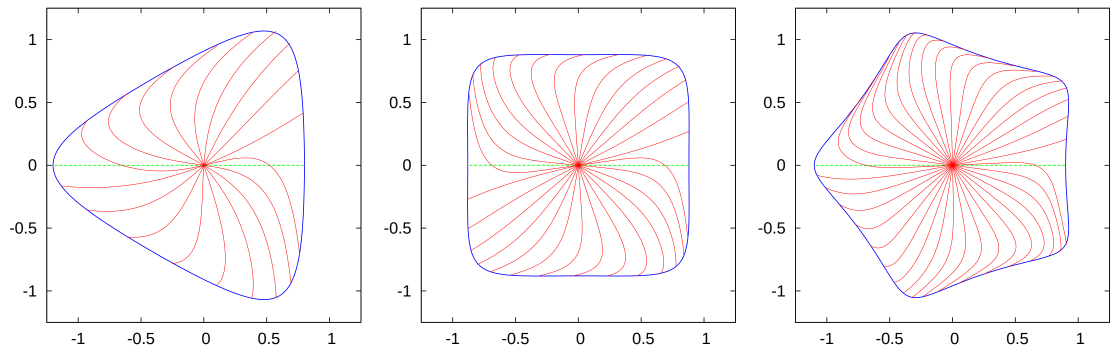
where $\lambda := \Omega/(\omega/2)$ and $\boldsymbol{\tau}$ is a vector tangent to the vortex boundary. Note that, despite its simple form, the problem (2) is a strongly nonlinear one, the integration curve in the Cauchy integral being unknown, as well as its density $\boldsymbol{\Phi}$.



(a) (b) (c) (d)

Figure 1: Samples of vortex systems of kind I are drawn with red lines and symbols. The initial guesses are also drawn (blue), together with the center of vorticity of the system (black filled circle). In $(a,\, b)$ there are two point vortices and in $(c,\, d)$ four. The point vortices have the same circulation in $(a,\, c)$, while a vortex has circulation larger than the others in $(b,\, d)$.

The set of the solutions of the problem (2) can be divided in two large subsets. The first one is formed by Schwarz functions such that the sum in the square brackets of equation (2) vanishes, *i.e.* $(\lambda - 1/2)\boldsymbol{\Phi} + \boldsymbol{C}(\boldsymbol{\Phi}) - \boldsymbol{F} = 0$. In these conditions, the angular speed Ω takes the value $\omega/2$, so that the vortex system is *in solid body rotation*. Samples of vortex systems of this kind (I, hereafter), obtained by means of an iterative procedure, are shown in Fig. 1 by starting from symmetric $(a,\, c)$ and unsymmetric $(b,\, d)$ initial guesses.



(a) (b) (c)

Figure 2: Approximate symmetric solutions of the equation (2) of kind II, for $n = 3$ (a, period of rotation T, with $\omega T \simeq 20.053$), $n = 4$ (b, $\omega T \simeq 17.252$) and $n = 5$ (c, $\omega T \simeq 16.111$). At time $t = 0$, a set of particles is placed on the dashed green line inside each vortex. Their relative positions at successive times (time step $T/10$, counterclockwise) are also drawn with red lines.

The second subset of solutions of the problem (2) is formed by Schwarz functions $\boldsymbol{\Phi}$ for which $(\lambda - 1/2)\boldsymbol{\Phi} + \boldsymbol{C}(\boldsymbol{\Phi}) - \boldsymbol{F} \neq 0$, so that the constraint on the real part becomes effective. To the best of the author knowledge, the only known solution belonging to this subset (of kind II, hereafter) is the Kirchhoff elliptical vortex, even if a bifurcation analysis shows that “an infinite family of rotating vortices with n-polygonal symmetry exists” [2] (the elliptical vortex is obtained for $n = 2$). The relation (1) allows for an easy calculation of approximate solutions of this family: some of them are shown in Fig. 2.

The relation (1) has been also applied for investigating the dynamics of simple vortex systems, together with the evolution equation of $\mathbf{\Phi}$:

$$D_t \mathbf{\Phi} = (\partial_t + \boldsymbol{u} \partial_{\boldsymbol{x}}) \mathbf{\Phi} = \overline{\boldsymbol{u}} \, .$$

A strongly nonlinear, singular integrodifferential problem in $\mathbf{\Phi}$ is obtained. By adopting the Lagrangian description of the motion and separing the singular linear term from the regular nonlinear one, the problem is analytically handled by means of successive approximations. However, due to the strongly increasing algebraic difficulties, only the solutions at orders 0 (nonlinear term set to 0) and 1 (nonlinear term evaluated by means of the 0-order approximation) have been computed.

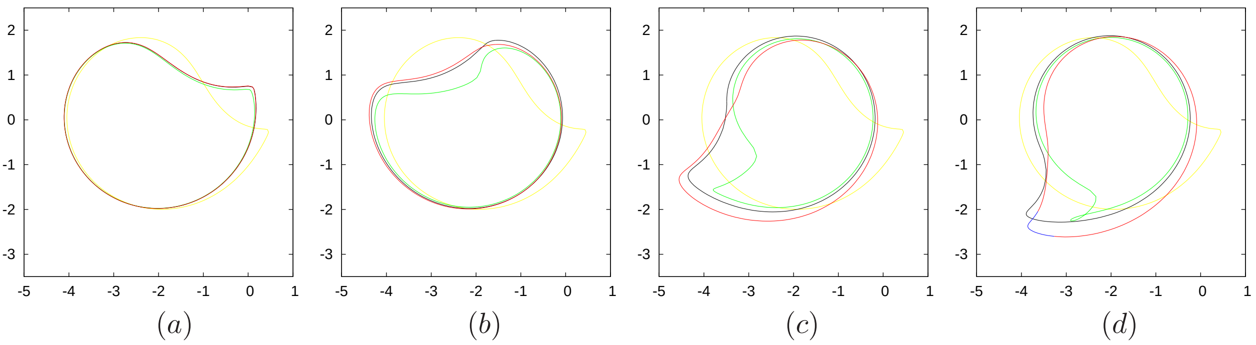


Figure 3: Approximate solutions at order 0 (green line) and 1 (red) at four consecutive times (a, b, c, d) for the self-induced motion of a uniform vortex. The initial condition is drawn with a yellow line, while the result of the numerical simulation of the flow are drawn with a black line. Note that the 0th order solution becomes non-simple in (d).

The self-induced motion of a uniform vortex has been investigated in the first place [8, 9]. The sample case in Fig. 3 demonstrates that the self-induced dynamics of the vortex is well captured by the 1st order approximation, while the 0th order solution breaks shortly. Encouraged by these results, the analysis has been extended to the interactions betweeen uniform and pointwise vortices [10]. The comparison with the numerical simulations performed for the sample cases in Fig. 4 proves that the 1st order solution is also able to reproduce the deformations of the vortex core induced by the point vortex, at least for small times. The motion of the point vortex appears mainly related to linear terms, being satisfactory described at order 0.

3 Conclusions and future work

Some applications of a novel integral relation between the Schwarz function of the boundary of a uniform vortex and the induced velocity have been discussed. The use of such relation together with classical tools of the Complex Analysis leads to new insigths into vortex equilibrium and dynamics.

In particular, asymmetric equilibrium configurations of vortex systems of kind I are easily numerically found and can be now investigated in an analytical fashion. The research of the steady vortices of kind II is much more difficult, fully involving the constraint on the real part (2). At the present time, only approximate solutions with n-polygonal symmetry (shown in Fig. 2) have been found, while the improvement of these approximations is under investigation.

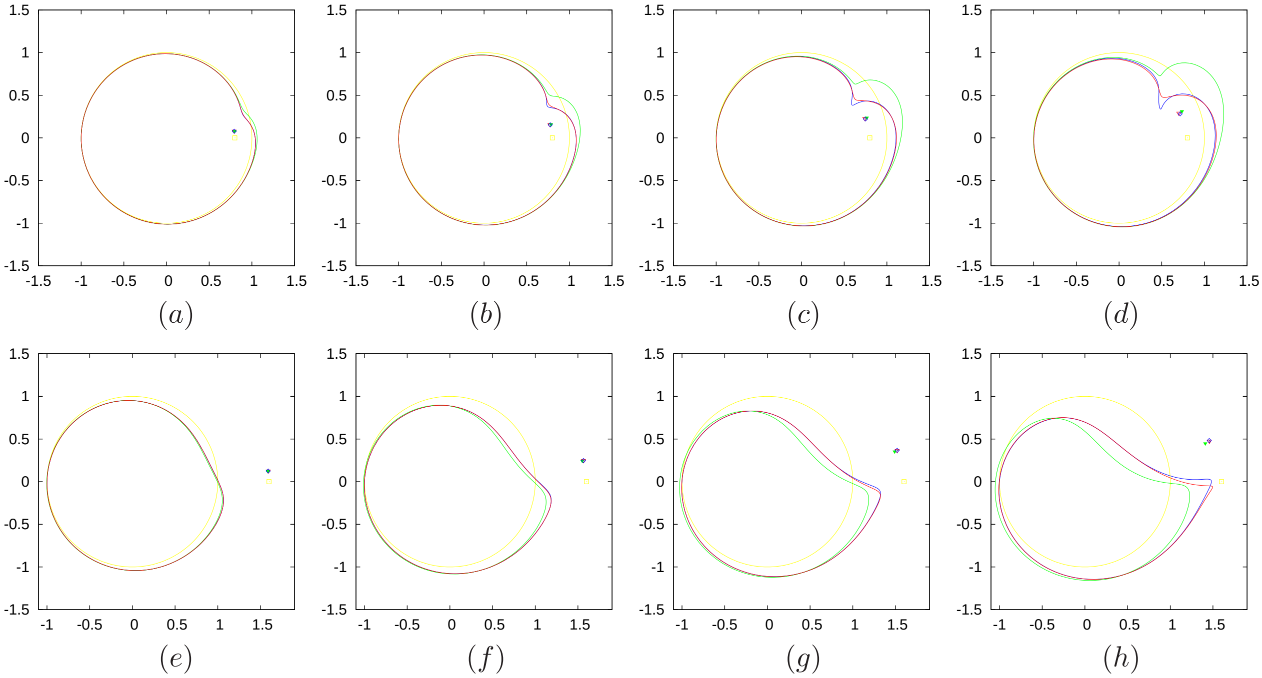


Figure 4: As in Fig. 3, but for the interaction of a uniform vortex and a pointwise one. The point vortex lies inside P in the first row (a, b, c, d) and outside in the second (e, f, g, h).

The motion of simple vortex systems has been also investigated by means of the Lagrangian representation of the flow and successive approximations of orders 0 and 1. In particular, the 1st order approximation accounts for nonlinearities at the lowest level and describes the vortex dynamics in terms of changes in time of the singular points of the Schwarz function. This leads to the conjecture that the same can be valid in general, one of the major issues under investigation at the present time. Sample cases of self-induced motion and interaction with a point vortex have been compared with numerical simulations of the flow. The 1st order approximation appears to be able to follow the dynamics, at least for short times. For this reason it will be extended to the analysis of the motion of two uniform vortices, and in particular of their merging.

References

[1] Jimenez, J.: Linear stability of a non-symmetric, inviscid Karman street of small uniform vortices. Journal of Fluid Mechanics 189, 337 – 348 (1988)

[2] Saffman, P.G.: Vortex Dynamics. Cambridge Univ. Press (1992)

[3] Crowdy, D.G.: The construction of exact multipolar equilibria of the two-dimensional Euler equations. Physics of Fluids 14-1, 257 – 267 (2001)

[4] Crowdy, D.G.: Exact solutions for rotating vortex arrays with finite-area cores. Journal of Fluid Mechanics 469, 209 – 235 (2002)

[5] Crowdy, D.G., Marshall, J.: Analytical solutions for rotating vortex arrays involving multiple vortex patches. Journal of Fluid Mechanics 523, 307 – 337 (2005)

[6] Riccardi, G.: Intrinsic dynamics of the boundary of a two-dimensional uniform vortex. Journal of Engineering Mathematics 50-1, $51 - 74$ (2004)

[7] Riccardi, G., Durante, D.: Velocity induced by a plane uniform vortex having the Schwarz function of its boundary with two simple poles, Journal of Applied Mathematics, Hindawi Pub. Corporation (2008)

[8] Riccardi, G.: Toward analytical contour dynamics. In: De Bernardis, E., Spigler, R., Valente, V. (eds.) Applied and Industrial Mathematics in Italy *III* (Advances in Mathematics for Applied Sciencies - vol. 82) pp. $496 - 507$. World Scientific, Singapore (2009)

[9] Riccardi, G.: An analytical study of the self-induced inviscid dynamics of two-dimensional uniform vortices. Acta Mechanica 224-2, $307 - 326$ (2013)

[10] Riccardi, G.: Initial stages of the interaction between uniform and pointwise vortices in an inviscid fluid, European Journal of Mechanics/B Fluids 53, $160 - 170$ (2015)