Envelope equation for gravity waves on deep water

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We consider a "unidirectional" motion of weakly nonlinear gravity waves on the surface of deep water, i.e. we assume that the spectrum of the free surface contains only nonnegative wavenumbers. We use remarkably simple form of the water wave equation that we named "the super compact equation" [1]. This equation has also very important property. It allows to introduce exact envelope for waves without assumption of narrowness bandwidth.

We introduce envelope function for waves, capital $C(x,t)$(we suppose that Fourier spectrum of waves has a maximum at $k=k_0$):

$$c(x,t) = C(x,t)e^{i(k_0 x - \omega_{k_0} t)} \tag{1}$$

For the envelope (1) one can easily derive the exact equation without assumption of narrow bandwidth for C:

$$\frac{\partial C}{\partial t} + i\hat{\mathcal{D}}_k^{(2)} C + ik_0^2|C|^2 C + 3k_0|C|^2\frac{\partial C}{\partial x} + k_0 C^2 \frac{\partial C^*}{\partial x} - ik_0 \mathcal{U} C - \frac{\partial}{\partial x}(i|C|^2\frac{\partial C}{\partial x} + \mathcal{U}C) = 0 \tag{2}$$

Here, velocity $\mathcal{U} = \hat{k}|C|^2$ and $\hat{\mathcal{D}}_k^{(2)} = \left[\hat{\omega}_{k_0+k} - \omega_{k_0} - \frac{\partial \omega_{k_0}}{\partial k_0}\hat{k}\right]$. Here operators $\hat{\omega}_{k_0+k}$ and $\hat{k}$ acts in k-space, so that $\hat{\omega}_{k_0+k} \to \sqrt{g(k_0+k)}$ and $\hat{k} \to |k|$. One can extract the NLSE and the Dysthe [2] equations from (2).

Physical variables $\eta(x,t)$ and $\psi(x,t)$ can be recovered from complex variable $c(x,t)$ by using canonical transformation. (Here we present only the linear and second order terms for $\eta(x,t)$):

$$\eta(x) = \frac{1}{\sqrt{2}g^{\frac{1}{4}}}(\hat{k}^{-\frac{1}{4}}c(x) + \hat{k}^{-\frac{1}{4}}c(x)^*) + \frac{\hat{k}}{4\sqrt{g}}[\hat{k}^{-\frac{1}{4}}c(x) - \hat{k}^{-\frac{1}{4}}c^*(x)]^2 \tag{3}$$

Operators $\hat{k}^\alpha$ act in Fourier space as multiplication by $|k|^\alpha$.

The equation (2) is very suitable for numerical simulations, which are not more complicated than numerical simulations in the framework of Nonlinear Schrodinger equation.

Being unavoidable phenomenon on the surface of the ocean, wave breaking is very difficult to describe or simulate. We performed numerical simulations to study in detail initial stage of wave-breaking in the framework of equation (2). We propose simple model of dissipation of such waves. This dissipation acts only in the point of breaking and does not affect other part of water surface. Numerical simulation shows reasonable behavior of integrals of motion, they drop by fixed value. After each event of this model breaking the surface again becomes smooth.

Using proposed simple model of dissipation we performed numerical simulation of long time evolution of initially slightly perturbed uniform wave train with wavelength 100m and steepness of waves $\mu \sim 0.1$. Numerical simulation have shown decay of homogeneous wave train into set of solitons which have different velocities and approximately the same width. They collide exchanging the energy. After a long time ($\sim$ 1.5 days) we observed a several large solitons which obviously took energy from the weaker. These solitons are very narrow, there are about three waves under the envelope (three sisters).

Acknowledgements This work was supported by Grant "Wave turbulence: theory, numerical simulation, experiment"#14-22-00174 of Russian Science Foundation.

References

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